REMARKS ON A QUASI-LINEAR MODEL OF THE NAVIER-STOKES EQUATIONS

FABIAN WALEFFE

Abstract. Dinaburg and Sinai recently proposed a quasi-linear model of the Navier-Stokes equations. Their model assumes that nonlocal interactions in Fourier space are dominant, contrary to the Kolmogorov turbulence phenomenology where local interactions prevail. Their equation corresponds to the linear evolution of small scales on a background field with uniform gradient, but the latter is defined as the linear superposition of all the small scale gradients at the origin. This is not self-consistent.

Dinaburg and Sinai [1] recently proposed a quasi-linear approximation of the Navier-Stokes equations which they feel preserves the basic character of the Navier-Stokes nonlinearity. They prove existence and uniqueness of solutions to their model, for special cases [2]. Here, we show that their equation is identical to that governing the linear evolution of small scales on an infinitely large scale flow with uniform gradient (eqn. (11) below). This is a direct consequence of their assumption that nonlocal interactions in Fourier space dominate. That assumption is contrary to a large body of phenomenological, experimental and numerical studies of the turbulent energy cascade where local interactions are thought to dominate (see *e.g.* [3]). On that basis alone, it seems unlikely that the model would preserve the basic character of the Navier-Stokes nonlinearity, but the model also contains a basic inconsistency. It defines the velocity gradient of the large scale field as the net velocity gradient induced by all the small scales at the origin (eqn. (12) below). This is not self-consistent since small scales do not uniformly distort larger scales. A self-consistent mean field theory, where the large scale flow results from the nonlinear interactions of the small scale fluctuations, instead of their linear superposition, is not possible in the nonlocal limit.

Consider the Navier-Stokes equations for incompressible flow of a viscous fluid in three-dimensional space $\mathbb{R}^3$:

$$\begin{aligned} \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p &= \nu \nabla^2 u, \\ \nabla \cdot u &= 0, \end{aligned} \tag{1}$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the velocity vector field, $p = p(x,t)$ is the kinematic pressure, $x = (x_1, x_2, x_3)$ is the position vector and $\nu > 0$ is the kinematic viscosity.

Let $u = A \cdot x + v$ where $A = A(t)$ is a matrix in $\mathbb{R}^3 \times \mathbb{R}^3$ such that $dA/dt + A \cdot A$ is symmetric and $\mathrm{tr}(A) = 0$ in order for $A \cdot x$ to be a solution of the incompressible Navier-Stokes equations (1). The base field $A \cdot x$ has uniform gradient and $v = v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t))$ is a velocity perturbation.

Date: February 1, 2008.
Key words and phrases. Navier-Stokes equations.

Substituting $u = A\cdot x + v$ in (1) and linearizing in v (*i.e.* omitting $(v\cdot\nabla)v$) yields

$$\frac{\partial v}{\partial t} + \Big((A\cdot x)\cdot\nabla\Big)v + A\cdot v + \nabla p = \nu\nabla^2 v, \qquad \nabla\cdot v = 0, \tag{2}$$

or, in indicial notation,

$$\frac{\partial v_l}{\partial t} + A_{mn}x_n\frac{\partial v_l}{\partial x_m} + A_{lm}v_m + \frac{\partial p}{\partial x_l} = \nu\frac{\partial^2 v_l}{\partial x_m\partial x_m}, \qquad \frac{\partial v_l}{\partial x_l} = 0. \tag{3}$$

We will use the convention of summation over repeated indices $l, m, n = (1, 2, 3)$. The self-advection of the background field is the gradient of a potential that has been absorbed into the pressure, since $dA/dt + A\cdot A$ is symmetric. Incompressibility of the base field requires $\mathrm{tr}(A) = A_{ll} = 0$. An equation for the pressure can be derived, in a standard manner, by taking the divergence of the momentum equation in (3) and using incompressibility to obtain

$$\nabla^2 p = \frac{\partial^2 p}{\partial x_l\partial x_l} = -2A_{ml}\frac{\partial v_l}{\partial x_m}. \tag{4}$$

Linear problems of the form (2) have been considered by many authors since Kelvin. There has been a renewed interest in such analyses in more recent times linked to a simple model of the elliptical instability (see [4] and references therein).

Kelvin modes. Kelvin noticed that a generalized Fourier analysis could be used to solve equation (2). He proposed to look for solutions of the form

$$v(x,t) = \hat{v}(t)e^{ik(t)\cdot x} \tag{5}$$

i.e. Fourier modes with time-dependent wave-vectors $k \in \mathbb{R}^3$. The motivation for this ansatz is that a Fourier mode initial condition proportional to $e^{ik\cdot x}$ is rotated and stretched uniformly by the background field with uniform gradients. Hence, it remains in the form of a Fourier mode, albeit with an evolving wavevector k. Substituting (5) together with $p(x,t) = \hat{p}(t)e^{ik(t)\cdot x}$ into (2), and using (4) to eliminate the pressure, leads to the coupled ordinary differential equations:

$$\frac{dk}{dt} = -k\cdot A, \tag{6}$$

$$\frac{d\hat{v}}{dt} = -\nu|k|^2\hat{v} - A\cdot\hat{v} + 2\frac{k}{|k|^2}\left(k\cdot A\cdot v\right). \tag{7}$$

The incompressibility constraint $\nabla\cdot v = 0$ requires $k(t)\cdot\hat{v}(t) = 0$. This is satisfied automatically since the pressure is determined from (4), provided that the initial conditions are such that $k(0)\cdot\hat{v}(0) = 0$. The most interesting solutions of these equations, perhaps, occur when the base flow $A\cdot x$ has closed streamlines. Then $k(t)$ is oscillatory and $\hat{v}(t)$ can grow exponentially through a parametric instability [4].

Fourier transform. Alternatively, one can proceed with a direct Fourier transform of the equations. Let $\hat{v}(k,t)$ be the Fourier transform of the vector

$$v(x,t) = \int_{\mathbb{R}^3}\hat{v}(k,t)\,e^{ik\cdot x}dk, \tag{8}$$

then

$$A_{mn} x_n \frac{\partial v_l(x,t)}{\partial x_m} = -\int_{\mathbb{R}^3} A_{mn} \frac{\partial}{\partial k_n}\Big(k_m \hat{v}_l(k,t)\Big) e^{ik\cdot x} dk, \tag{9}$$

which, in our case, simplifies to

$$A_{mn} x_n \frac{\partial v_l(x,t)}{\partial x_m} = -\int_{\mathbb{R}^3} k_m A_{mn} \frac{\partial \hat{v}_l(k,t)}{\partial k_n} e^{ik\cdot x} dk, \tag{10}$$

because $A_{mm} = 0$. It follows that the Fourier transform of equation (3) reads

$$\frac{\partial \hat{v}_l(k,t)}{\partial t} - k_m A_{mn} \frac{\partial \hat{v}_l}{\partial k_n} = -\nu |k|^2 \hat{v}_l - A_{lm} \hat{v}_m + 2\frac{k_l}{|k|^2}\big(k_m A_{mn} \hat{v}_n\big), \tag{11}$$

where the pressure and the incompressibility constraint have been eliminated using (4) provided the initial conditions satisfy $k_l \hat{v}_l(k,0) = 0$, $\forall k$. Solving equation (11) by the method of characteristics, we directly recover Kelvin modes and equations (6), (7).

Equation (11) is identical to Dinaburg and Sinai's equation (11) except for a different definition of the matrix A, theirs is minus the transpose of ours: $A^{(DS)} = -A^T$. Note that equation (11) has been obtained without imposing the point symmetry $v(x,t) = -v(-x,t)$, assumed by Dinaburg and Sinai, although the base flow $A \cdot x$ does satisfy that symmetry. The point symmetry implies that $\hat{v}(k,t) = -\hat{v}(-k,t)$. Now, $\hat{v}(k,t) = \overline{\hat{v}(-k,t)}$ since $v(x,t)$ is real, where the overline denotes complex conjugate. Hence, $\hat{v}(k,t)$ must be pure imaginary if the point symmetry is imposed. So Dinaburg and Sinai's $v(k,t)$ is related to our $\hat{v}(k,t)$ as $v^{(DS)}(k,t) = -i\hat{v}(k,t)$.

Dinaburg and Sinai define

$$A^{(DS)}_{mn} = -\int i k_m \hat{v}_n(k,t) dk = -\left.\frac{\partial v_n(x,t)}{\partial x_m}\right|_{x=0}, \tag{12}$$

while, if we write our base flow as $U(x,t) = A(t)\cdot x$, then $U_m = A_{mn} x_n$ and

$$A_{mn} = \frac{\partial U_m(x,t)}{\partial x_n}. \tag{13}$$

This explains the differences in the definitions of the velocity gradient A. It also points to a basic inconsistency of the Dinaburg-Sinai model. Equation (11) corresponds to the distortion of small scales by an infinitely large scale flow with uniform gradient, but Dinaburg and Sinai define the gradient of the infinitely large scale velocity field as the local gradient at $x = 0$ resulting from the linear superposition of all the small scale gradients. Considering two distinct Fourier modes with wavevectors $k^{(1)}$ and $k^{(2)}$, for instance, with $|k^{(1)}| < |k^{(2)}|$, it does not make sense to have the small scale $k^{(2)}$ participating in the uniform large scale distortion of the larger scale $k^{(1)}$. Furthermore, the mode $k^{(1)}$ does not self-distort because of the incompressibility constraint (so $(v \cdot \nabla)v = 0$ for a single Fourier mode). Hence, only larger scales should contribute to the approximation of uniform distortion of a given small scale. In other words, only $|k'| \ll |k|$ should contribute to the large scale gradient distorting the mode with Fourier wavevector k. This can be seen also at a technical level in the derivation of the Dinaburg-Sinai model. In their treatment of the convolution integral representing the Fourier transform of the Navier-Stokes nonlinearity, Dinaburg and Sinai make the assumption that the integral is dominated by highly nonlocal interactions, *i.e.* by the domains $|k'| \ll |k|$

and $|k-k'| \ll |k|$. Considering the domain $|k'| \ll |k|$, they make the following type of approximation, for instance,

$$\int_{\mathbb{R}^3} k'_n u_m(k') u_n(k-k') dk' \approx u_n(k) \int k'_n u_m(k') dk' \tag{14}$$

(see eqn. (6) in [1]). The domain of integration for the integral on the right-hand side should be restricted to the ball $B_\epsilon = \{k' \in \mathbb{R}^3 : |k'| < \epsilon\}$, with $\epsilon \ll |k|$, for self-consistency, but Dinaburg and Sinai do not specify the domain of integration. In their later studies of finite dimensional approximations, they sum over all modes, in other words, they integrate over $\mathbb{R}^3$ instead of B_ϵ. If the integral, and the finite dimensional sums, were correctly restricted, the model would not be closed, or the infinitely large scale flow would have to be specified and the model would become linear and identical to (11), with $A(t)$ specified independently from the small scales.

It is natural to wonder whether one could replace the Dinaburg-Sinai model by a mean field theory where the large scale flow results from the nonlinear interactions of the small scales. The base field $U = A \cdot x$ is a very singular $k = 0$ mode whose generalized Fourier transform $\hat{U}(k) = i(A \cdot \nabla_k)\delta(k)$. Here $\delta(k) = \delta(k_1)\delta(k_2)\delta(k_3)$ is a product of Dirac delta functions and ∇_k is the gradient operator in k-space with $\partial\delta(k)/\partial k_1 = \delta'(k_1)\delta(k_2)\delta(k_3)$, where $\delta'(k_1)$ is the generalized derivative of the delta function, and similarly for derivatives with respect to k_2 and k_3. The only nonlinear interactions that can create a $k = 0$ mode consist of any mode k' interacting with its complex conjugate $-k'$, but such interactions vanish because of incompressibility. For instance, the convolution integral on the left-hand side of (14) when $k = 0$,

$$\int_{\mathbb{R}^3} k'_n u_m(k') u_n(-k') dk' = 0 \tag{15}$$

for any regular $u(k')$ because $k'_n u_n(-k') = 0$ from incompressibility. The vanishing of such interactions is also related to Galilean invariance since a $k = 0$ mode could also correspond to a constant velocity.

References

[1] E.I. Dinaburg & Ya.G. Sinai, "A quasilinear approximation for the three-dimensional Navier-Stokes system," *Moscow Math. J.* **1**, (3) 381-388 (2001).

[2] E.I. Dinaburg & Ya.G. Sinai, "Existence and Uniqueness of Solutions of a Quasilinear Approximation of the 3D Navier-Stokes System," *Problems of Information Transmission* **39**, (1) 47-50 (2003).

[3] U. Frisch, "Turbulence: The Legacy of A.N. Kolmogorov," Cambridge University Press, Cambridge, 1995.

[4] R.R. Kerswell, "Elliptical Instability," *Annu. Rev. Fluid Mech.*, **33**, 83-113 (2002).

Department of Mathematics, University of Wisconsin, Madison, WI 53706
E-mail address: waleffe@math.wisc.edu